

# ASYMPTOTICALLY HARMONIC MANIFOLDS WITH MINIMAL HOROSPHERES

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*Hemangi Shah would like to dedicate this article to her father, Shri Madhusudan Shah, to ignite a spark of creativity in her.*

**ABSTRACT.** Let  $(M, g)$  be a complete Riemannian manifold without conjugate points. In this paper, we show that if  $M$  is also simply connected, then  $M$  is flat, provided that  $M$  is asymptotically harmonic manifold with minimal horospheres (AHM). Consequently, we conclude that AH manifold of subexponential volume growth are of polynomial volume growth. Thus AH manifolds can have either polynomial or exponential volume growth.

## 1. INTRODUCTION AND PRELIMINARIES

One of the important problems in the geometry of Hadamard manifold (manifold of nonpositive curvature)  $M$  is: To what extent *horospheres* (geodesic sphere of infinite radius) of  $M$  determine the geometry of  $M$ ? This is a very hot topic of current research as can be seen from the very recent paper [8], where the connection between hessian of Busemann functions and rank of Hadamard manifold is established.

Horospheres, geodesic spheres of infinite radius, of  $M$ , are level sets of *Busemann function*, which is defined as follows. Let  $M$  be a complete, simply connected Riemannian manifold without conjugate points. Then by Cartan-Hadamard theorem, every geodesic of  $M$  is a line. Let  $SM$  be the unit tangent bundle of  $M$ . For  $v \in SM$ , let  $\gamma_v$  be the geodesic line in  $M$  with  $\gamma'_v(0) = v$ . Then  $b_v^+ : M \rightarrow \mathbb{R}$  and  $b_v^- : M \rightarrow \mathbb{R}$  denote two Busemann functions associated to  $\gamma_v$ , respectively, towards  $+\infty$  and  $-\infty$ , and are defined as:

$$\begin{aligned} b_v^+(x) &= \lim_{t \rightarrow \infty} d(x, \gamma_v(t)) - t, \\ b_v^-(x) &= \lim_{t \rightarrow -\infty} d(x, \gamma_v(t)) + t. \end{aligned}$$

Note that  $b_v^\pm(\gamma_v(r)) = \mp r$ . The level sets,  $b_v^{\pm-1}(t)$ , are called *horospheres* of  $M$ . Thus, two Busemann function can be interpreted as distance function from  $\pm\infty$ , can be defined for any line in  $M$ . Refer [19] for details on

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Busemann functions.

A complete Riemannian manifold is called *harmonic*, if all the geodesic spheres of sufficiently small radii are of constant mean curvature. The known examples of harmonic manifolds include flat spaces and locally rank one symmetric spaces. In 1944, Lichnerowicz conjectured that any simply connected harmonic manifold is either flat or a rank one symmetric space. This conjecture is true in the compact case and false in the non-compact one. However, the other questions in the non-compact case remain open. For the development of the conjecture see the references in [23]. In particular, if an harmonic manifold does not have conjugate points then the geodesic spheres of any radii are of constant mean curvature and is also AH (cf. [7, Remark 2.2], cf. [21]). Moreover, by Allamigeon's Theorem [2], it follows that any complete, simply connected and non-compact harmonic manifold has no conjugate points. For definition and more details about harmonic manifolds see [2]. In [7] J. Heber obtained the complete classification of simply connected harmonic manifolds in the homogeneous case.

It is very well known that, eg., real hyperbolic space,  $\mathbb{H}^n$ , real space form, is characterized by its horospheres. Equivalently,  $\mathbb{H}^n$ , which is a *harmonic manifold*, is characterized by its volume density function, denoted by  $\Theta(r)$ , which in polar co-ordinates can be expressed as  $\Theta(r) = \sqrt{\det g_{ij}}(r)$ . In this case,  $\Theta(r) = \sinh^{n-1}(r)$  and the mean curvature of horospheres  $\frac{\Theta'(r)}{\Theta(r)} = (n-1) = \Delta b_r^\pm$ .

Szabo asked: To what extent the density function of a harmonic manifold  $M$ , determine the geometry of  $M$ . The affirmative answer to his question can be found in [20], in case of rank one harmonic symmetric spaces.

*Asymptotically harmonic* manifolds are asymptotic generalization of *harmonic* manifolds. A complete, simply connected Riemannian manifold without conjugate points is called AH, if the mean curvature of its horospheres is a universal constant. Equivalently, if its Busemann functions satisfy  $\Delta b_v^\pm \equiv h$ ,  $\forall v \in SM$ , where  $h$  is a nonnegative constant (see [7]).

Till to-date the only known examples of AH manifolds are harmonic manifolds. Intuitively one thinks that the class of harmonic and AH manifolds coincide. And the Lichnerowicz conjecture should be true for AH manifolds as well. In this paper we answer this question partially that these two classes coincide in case when  $h = 0$ . For more details on questions related to Lichnerowicz conjecture for AH manifolds see [29], [30] and [12].

In the sequel, AH manifold  $M$  is simply connected, complete Riemannian manifold without conjugate points and  $\Delta b_v^\pm \equiv h$ , for all  $v \in SM$ , where  $h$  is a nonnegative constant. Then by regularity of elliptic partial differential equations,  $b_v^\pm$  is a smooth function on  $M$  for all  $v$  and all horospheres of  $M$  are smooth, simply connected hypersurfaces in  $M$  with constant mean

curvature  $h \geq 0$ . For example, every simply connected, complete harmonic manifold without conjugate points is asymptotically harmonic. See [21] for a proof.

Hence, on an asymptotically harmonic manifold  $M$  we can define  $(1, 1)$  tensor fields  $u^+$  and  $u^-$  as follows: For  $v \in SM$  and  $x \in v^\perp$ , let

$$u^+(v)(x) = \nabla_x \nabla b_{-v}^+ \quad \text{and} \quad u^-(v)(x) = -\nabla_x \nabla b_v^+.$$

Thus,  $u^\pm(v) \in \text{End}(v^\perp)$  and  $\text{tr } u^+(v) = \Delta b_{-v}^+ = h$ ,  $\text{tr } u^-(v) = -\Delta b_v^+ = -h$ .  $u^+(v), u^-(v)$ , respectively, is the second fundamental form of the *unstable and stable horosphere*  $b_v^{-1}(0), b_v^{+1}(0)$ , respectively, at  $p$ .

Moreover, the endomorphism fields  $u^\pm$  satisfy the Riccati equation along the orbits of the geodesic flow  $\varphi^t : SM \rightarrow SM$ . Thus, if  $u^\pm(t) := u^\pm(\varphi^t v)$  and the Jacobi operator,  $R(t) := R(\cdot, \gamma'_v(t))\gamma'_v(t) \in \text{End}(\gamma'_v(t)^\perp)$ , then

$$(1) \quad (u^\pm)'(t) + (u^\pm)^2(t) + R(t) = 0.$$

$u^+(t)$  and  $u^-(t)$  are called as *unstable and stable* Riccati solution respectively.

Now we explain the term *unstable and stable* used in the above context.

**Asymptote :** Let  $(M, g)$  be a complete, non-compact Riemannian manifold. Let  $\gamma_v$  be a ray. A vector  $w \in S_p M, p \in M$  arbitrary, is called an asymptotic to  $v$  if  $\nabla b_{v, s_i}(p) \rightarrow w$  for some sequence  $s_i \rightarrow \infty$ . Clearly  $\gamma_w$  is also a ray. We also say that  $\gamma_w$  is an asymptotic to  $\gamma_v$  from  $p$ .

**Bi-asymptote :** We say that two geodesics are bi-asymptotic, if they are asymptotic to each other in both the positive and negative directions. If all the asymptotic geodesics of  $M$  are bi-asymptotic, then we say that  $M$  has the *bi-asymptotic property*.

The proof of the Proposition 1.1 below is easy and hence we omit it.

**Proposition 1.1.** *Let  $(M, g)$  be a complete, simply connected Riemannian manifold without conjugate points. Let  $J$  be the Jacobi field arising out of variation of asymptotic geodesics. Then  $J$  satisfies  $u^+(t)(J(t)) = -J'(t), u^-(t)(J(t)) = J'(t)$ , for all  $t \in \mathbb{R}$ .*

*Remark 1.2.* Note that in case of Hadamard manifold of strictly pinched negative curvature, the Jacobi field described above is bounded as  $t \rightarrow +\infty$  and is unbounded as  $t \rightarrow -\infty$ . Hence, we call  $J$  as stable Jacobi field towards  $+\infty$  and unstable towards  $-\infty$ . And the corresponding horospheres are called as stable and unstable horospheres. Thus, in general when manifold is simply connected and without conjugate points, we call  $b_v^{-1}(0), b_v^{+1}(0)$  as unstable and stable horospheres, and  $J$  is called as unstable and stable Jacobi field corresponding to unstable and stable horospheres, in analogy with manifolds of negative curvature. If  $J$  is both unstable and stable Jacobi field, then it is called as central Jacobi field. See [30] for more details on this topic.

Note that study of AH manifolds starts from dimension 2. In dimension 2, the only AH manifolds are symmetric viz.,  $\mathbb{H}^2$  and  $\mathbb{R}^2$ . In dimension 3, AH manifolds have been classified in [6, 13, 24, 26] and in the non-compact, Einstein and homogeneous case in [7]. For more details on AH spaces, we refer to the discussion and to the references in [6]. Important results in this context are contained in [3] and [13].

Thus, AH manifolds are defined via the property that all of its horospheres, a parallel family of hypersurfaces have the same constant mean curvature. In view, of the above classification the natural analogue of Szabo's question is: whether the asymptotic growth of  $\Theta_p(r)$  or equivalently whether the geometry of horospheres of an AH manifold influence geometry of its ambient space?

For  $v \in SM$ , the corresponding horosphere is *totally umbilical* with constant principal curvature  $h \in \mathbb{R}$  if and only if  $u^+(t) = h \text{id}_{\gamma'_v(t)^\perp}$ . Thus, in this case  $M$  is Einstein of constant non-positive curvature  $-h^2$ , as  $R(t) = -h^2 \text{id}_{\gamma'_v(t)^\perp}$  from (1). Thus, if  $h = 0$ , then  $u^+(t) \equiv 0$ , i.e. if every horosphere of an AH manifold is totally geodesic, then the ambient manifold is flat. See [9] for more rigidity results about AH manifolds.

Moreover, the classification of 3-dimensional AH manifolds follows from [24] and a recent paper [26]. In particular, in [26] the following result was proved:

**Theorem 1.3.** *Let  $(M, g)$  be a complete and simply connected Riemannian manifold of dimension 3 without conjugate points. If  $M$  is AH of constant  $h = 0$ , then  $M$  is flat.*

In case of harmonic manifolds flatness follows in all dimensions from [23] (see also [25]).

**Theorem 1.4.** *Let  $(M, g)$  be a complete and simply connected Riemannian manifold. If  $M$  is a non-compact HM, then  $M$  is flat.*

If a non-compact, homogeneous and Einstein manifold is AH, then J. Heber [7] showed that:

**Theorem 1.5.** *Let  $(M, g)$  be a complete, simply connected, non-compact manifold without conjugate points. If  $M$  is homogeneous, Einstein and AH manifold of constant  $h = 0$ , then  $M$  is flat.*

In this paper we generalize Heber's theorem, Theorem 1.5. More precisely, we prove that:

**Main Theorem 1.6.** *Let  $(M, g)$  be a complete and simply connected manifold without conjugate points. If  $M$  is an AH manifold of constant  $h = 0$ , then  $M$  is flat.*

Note that as any non-compact harmonic manifold is AH, we recover Theorem 1.4 from Main Theorem 1.6.

The paper is organized into 6 sections. In section 2, we show that compact AHM are flat. In section 3, we show that Einstein and AHM are flat. In section 4, we show that an AHM  $M$  admit Strong Liouville Type Property and Liouville Property. In section 5, we show that  $M$  admits a

non-trivial Killing vector field. Then using two different types of argument, we show that  $M$  is flat. In section 6, we show that flat strip theorem holds for  $M$  and we reprove the result of [26] in dimension 3 by a different method.

## 2. ASYMPTOTICALLY HARMONIC MANIFOLDS WITH MINIMAL HOROSPHERES ADMITTING COMPACT QUOTIENT

In this section, we show that an AHM,  $M$  admitting compact quotient is flat. Thus, we prove the flatness of  $M$  without no focal point assumption as in [30].

We first recall here definition of complete AH manifold and that of volume growth and volume growth entropy.

*Definition 2.1.* Let  $(M, g)$  be a complete Riemannian manifold without conjugate points. Let  $(\tilde{M}, g)$  be its universal Riemannian covering space. Then  $(M, g)$  is called AH, if there exists a constant  $h \geq 0$  such that the mean curvature of all horospheres in  $\tilde{M}$  is  $h$ .

*Definition 2.2.* A manifold  $M$  is said to have *exponential volume growth* if there exists  $c > 1$  such that  $\text{Vol}(B(p, r)) \geq c^r$  for large  $r$ .

*Definition 2.3.* If volume growth of  $M$  is not exponential, then  $M$  is said to have *subexponential volume growth*.

*Definition 2.4.* If  $\text{Vol}(B(p, r)) \leq Cr^n$  for large  $r$  and for some  $C$  and  $n > 0$ , then  $M$  is said to have *polynomial volume growth*.

*Definition 2.5.* Let  $(M, g)$  be a compact Riemannian manifold and let  $(\tilde{M}, g)$  be its universal Riemannian covering space. Choose  $p \in \tilde{M}$ . The volume entropy (or asymptotic volume growth)  $h_v = h_v(M, g)$  is defined as :

$$h_v = h_v(M, g) = \lim_{R \rightarrow \infty} \frac{\log(\text{Vol}_{\tilde{M}}(B_p(R)))}{R}.$$

Manning [17] showed that the limit above always exists and is independent of  $p \in \tilde{M}$ .

*Definition 2.6.* Let  $(M, g)$  be a connected, non-compact Riemannian manifold. Then the volume entropy  $h_{vol}(M)$  of  $M$  is defined as :

$$(2) \quad h_{vol}(M, g) = \lim_{R \rightarrow \infty} \sup \frac{\log(\text{Vol}(B_p(R)))}{R},$$

where  $B_r(p) \subset M$  is the open ball of radius  $R$  around  $p \in M$ .

Note that (2) doesn't depend on the choice of reference point  $p$  and  $h_{vol}(M, g)$  is therefore well defined.

In [30] the following Proposition was proved.

**Proposition 2.7.** *Let  $(M, g)$  be a compact AHM. If  $M$  has no focal points, then  $M$  is flat.*

We prove the above result without no focal point condition, mainly by using the following result of [30].

**Proposition 2.8.** *Let  $(M, g)$  be a compact, AH manifold with  $\tilde{M}$  as universal covering space of  $M$  and  $\Delta b_v = h$ , for all  $v \in S\tilde{M}$ . Then  $h = h_v$ .*

**Theorem 2.9. Tits Theorem:** *A finitely generated subgroup of a connected Lie group has either exponential growth or is almost nilpotent and hence has polynomial growth.*

**Theorem 2.10.** *Let  $(M, g)$  be a compact AHM. Then  $(M, g)$  is flat.*

*Proof.* Let  $\tilde{M}$  be the universal covering space of  $M$ . Then  $\Delta b_v \equiv 0$ , for all  $v \in S\tilde{M}$ . And hence the volume growth entropy  $h_v = 0$ , by Proposition 2.8 of [30]. Hence, volume growth of  $\tilde{M}$  is subexponential. Hence,  $\pi_1(M)$  is of subexponential growth. But by Tits' Theorem,  $\pi_1(M)$  has polynomial growth, as it is a subgroup of connected Lie group  $\text{Isom}(\tilde{M})$ , of subexponential growth. By [14] compact Riemannian manifold without conjugate points and with polynomial growth fundamental group are flat. Thus  $(M, g)$  is flat.  $\square$

As any harmonic manifold is AH we conclude:

**Corollary 2.11.** *Let  $(M, g)$  be a compact HM. Then  $M$  is flat.*

*Remark 2.12.* (1) Theorem 2.10 should be known, but the authors couldn't find the reference where it is explicitly proved and hence they include it here.

(2) In view of Theorem 2.10, it suffices to show that a non-compact AHM is flat. This is proved in the remaining sections.

### 3. ASYMPTOTICALLY HARMONIC AND EINSTEIN MANIFOLDS WITH MINIMAL HOROSPHERES

In this section, we show that AH manifolds which are Einstein are flat. Thus, we generalize Heber's result, Theorem 1.5, quoted in the section 1 of the paper. We also recover that a HM is flat. The original proof of the flatness of harmonic manifolds, Theorem 1.4, given in [23] is elegant but needs some complicated reasoning. Thus an alternative proof by a simpler method was presented in [25]. This section uses ideas from [25], although here the proof of [25, Lemma 3.4], viz. Lemma 3.5 is explained with more clarity.

From [29] we also know

**Lemma 3.1.** *If  $(M, g)$  is an AH, then the map  $v \rightarrow u^\pm(v)$  is continuous on  $SM$ . Consequently,  $v \rightarrow \lambda_i^\pm(v)$  are continuous functions, where  $\lambda_i^\pm(v)$  are eigenvalues of  $u^\pm(v)$ .*

**Lemma 3.2.** *Let  $(M, g)$  be an AHM. Let  $\gamma_v$  be a geodesic line, then  $b_v^+ + b_v^- = 0$ .*

*Proof.* Let  $\gamma_v$  be a geodesic line. As  $(M, g)$  is an AHM,  $\Delta b_v^\pm = h = 0$ . Also,

$$b_v^+(x) + b_v^-(x) = \lim_{t \rightarrow \infty} d(x, \gamma_v(t)) + d(x, \gamma_v(-t)) - 2t.$$

Hence,  $b_v^+(x) + b_v^-(x) \geq 0$  for all  $x$ , by triangle inequality, and  $(b_v^+ + b_v^-)(\gamma_v(t)) = 0$ , since  $\gamma_v$  is a line.

Thus, the minimum principle shows that  $b_v^+ + b_v^- = 0$ .  $\square$

*Remark 3.3.* The above lemma shows that the stable horospheres,  $(b_v^+)^{-1}(0)$ , and unstable horospheres,  $(b_v^-)^{-1}(0)$ , of an AHM coincide like flat spaces.

**Corollary 3.4.** *Let  $(M, g)$  be an AHM then  $u^+(v) = u^-(v)$  for all  $v \in SM$ . Consequently,  $u^+(-v) = -u^+(v)$*

*Proof.* From the definition of  $b_v^\pm$ , the equation  $b_v^+ + b_v^- = 0$  is equivalent to  $b_v^+(x) = -b_{-v}^+(x)$ . Hence,  $\nabla_x \nabla b_v^+ = -\nabla_x \nabla b_{-v}^+$ . Thus, from the definition of  $u^\pm$  we get,

$$(3) \quad u^+(v) = u^-(v), \quad \forall v \in SM$$

From definition of  $u^\pm(v)$  we have,  $u^+(-v) = -u^-(v)$ . Hence, equation (3) shows that

$$(4) \quad u^+(-v) = -u^+(v)$$

$\square$

**Lemma 3.5.** *Let  $(M, g)$  be an AHM and let  $p \in M$ . Then, there exist  $v_0 \in S_p M$  such that  $u^+(v_0) = 0$ . In particular, the Ricci curvature at  $v_0$  is zero i.e.  $\text{Ricci}(v_0, v_0) = 0$ .*

*Proof.* By Proposition 3.1, eigenvalues of  $u^+(v)$  vary continuously with  $v \in S_p M$ . Fix  $p \in M$ . Note that as dimension of  $M$  is  $n$ ,  $u^+(v)$  is  $(n-1) \times (n-1)$  traceless symmetric matrix. Hence, there exists a basis of  $v^\perp$  such that  $u^+(v)$  is represented by a diagonal matrix. Let  $\lambda_1^+(v), \lambda_2^+(v), \dots, \lambda_{n-1}^+(v)$  be eigenvalues of  $u^+(v)$ , such that

$$(5) \quad \lambda_1^+(v) \leq \lambda_2^+(v) \leq \dots \leq \lambda_{n-1}^+(v).$$

We may identify the tangent sphere  $S_p M$  with the standard  $(n-1)$ -sphere  $S^{n-1}$ . Now consider the continuous map  $f : S^{n-1} \rightarrow \mathbb{R}^{n-1}$  defined by

$$f(v) = (\lambda_1^+(v), \lambda_2^+(v), \dots, \lambda_{n-1}^+(v)).$$

Then by Borsuk-Ulam theorem there exists  $v_0 \in S^{n-1}$ , such that  $f(v_0) = f(-v_0)$ . Therefore,

$$(6) \quad \lambda_i^+(v) = \lambda_i^+(-v), \quad \forall i = 1, 2, \dots, (n-1).$$

But, equation (4) shows that operators  $u^+(v)$  and  $u^+(-v)$  commute for all  $v \in S_p M$ . Hence, (6) shows that

$$(7) \quad u^+(-v_0) = u^+(v_0)$$

Hence, equation (6) imply that  $u^+(v_0) = 0$ . Now  $\text{Ricci}(v_0, v_0) = 0$  follows from the Riccati equation.  $\square$

**Corollary 3.6.** *Let  $(M, g)$  be an AHM. If  $(M, g)$  is Einstein, then  $(M, g)$  is flat.*

*Proof.* If  $(M, g)$  is Einstein, then from Lemma 3.5,  $\text{Ricci} \equiv 0$  and  $M$  is Ricci flat. From, Riccati equation we get,

$$0 = \text{tr}(u^+)' + \text{tr}(u^+)^2 + \text{tr} R = \text{tr}(u^+)^2 = \|u^+\|^2.$$

Consequently,  $u^+(v) = 0$  for all  $v \in SM$ . Again Riccati equation shows that  $R(x, v)v = 0$ , for all  $x \in v^\perp$  and for all  $v \in SM$ . Thus  $M$  is flat.  $\square$

Thus, Corollary 3.6, generalizes Heber's theorem, Theorem 1.5. It is well known that any harmonic manifold is Einstein [2]. As harmonic manifolds are AH, we recover Theorem 1.4 in a more simpler way.

**Corollary 3.7.** *HM are Ricci flat and hence they are flat.*

Thus we reprove the following result which is generally affirmed by appealing to Cheeger-Gromov splitting theorem.

**Corollary 3.8.** *Ricci flat harmonic manifolds are flat.*

*Remark 3.9.* 1) Note that our method to prove Corollary 3.8 is entirely different than [20].

2) In view of results of this section, now it suffices to show that AH's are flat without compactness or Einstein assumption; which we show in the next sections.

#### 4. STRONG LIOUVILLE PROPERTY

In this section, we show that an AHM satisfy *Strong Liouville Type Property* and *Liouville Property*, in comparison with an HM, now we know them to be flat spaces.

*Definition 4.1.* A manifold  $M$  is said to satisfy Strong Liouville Type Property if, there are no non-trivial bounded subharmonic functions on  $M$ .

**Mean Value Inequality  $\mathcal{M}_R(\lambda)$  :** A manifold  $M$  is said to satisfy a mean value inequality  $\mathcal{M}_R(\lambda)$ , if there exists a constant  $\lambda > 0$  such that for any  $r \leq R$  and  $f(x) \geq 0$  satisfying  $\Delta f \geq 0$  on  $B_p(r)$ , then

$$f(p) \leq \frac{\lambda}{\text{Vol}(B_p(r))} \int_{B_p(r)} f(x) dx.$$

It is well known that all harmonic manifolds satisfy Mean Value Inequality  $\mathcal{M}_R(\lambda)$  with  $\lambda = 1$  for all  $R$ . See [28]

Now we recall the following theorem of Karp [11].

**Theorem 4.2.** *Let  $F(r)$  be any positive nondecreasing function satisfying  $\int_a^\infty \frac{dr}{rF(r)} = +\infty$  for some  $a > 0$ . If  $u$  is a non-constant, non-negative  $C^2$  solution of the inequality  $u\Delta u \geq 0$  on a complete non-compact Riemannian manifold  $M$ , then for every  $p > 1$  and  $x_0 \in M$  we have*

$$(8) \quad \lim_{r \rightarrow \infty} \sup \frac{1}{r^2 F(r)} \int_{B(x_0; r)} |u|^p d\text{vol} = +\infty.$$



**Proposition 4.3.** *Let  $(M, g)$  be an AH manifold of constant  $h \geq 0$ . Let  $\Theta_q(r)$  be the volume density function in polar co-ordinates centred at  $q \in M$ . Then*

$$\lim_{r \rightarrow \infty} \frac{1}{r^2 \Theta_q(r)} \int_0^r \Theta_q(r) = 0.$$

*Proof.*

$$\lim_{r \rightarrow \infty} \frac{1}{r^2 \Theta_q(r)} \int_0^r \Theta_q(r) = \lim_{r \rightarrow \infty} \frac{\Theta_q(r)}{2r \Theta_q(r) + r^2 \Theta_q'(r)} = 0,$$

$$\text{as } \lim_{r \rightarrow \infty} \frac{\Theta_q'(r)}{\Theta_q(r)} = h \geq 0. \quad \square$$

**Theorem 4.4.** *Let  $(M, g)$  be an AHM. Let  $u$  be a non-negative bounded subharmonic function on  $M$ , then  $u$  is a constant function on  $M$ .*

*Proof.* Let  $(M, g)$  be an AHM. Let  $\Theta_q(r)$  denote the volume density function in polar co-ordinates centred at  $q \in M$ . Since  $(M, g)$  is a simply connected, complete manifold without conjugate points,  $\Theta_q(r)$  is monotonically increasing function (cf. [21]) for any  $q \in M$ . Let  $F(r) = \Theta_q(r)$ . Then,

$$(9) \quad \int_1^\infty \frac{dr}{r F(r)} = \int_1^\infty \frac{dr}{r \Theta_q(r)} = \int_1^\infty \frac{dr}{r \Theta_q(r)} > \int_1^\infty \frac{dr}{r} = +\infty,$$

as  $(M, g)$  has minimal horospheres i.e.  $\lim_{r \rightarrow \infty} \frac{\Theta_q'(r)}{\Theta_q(r)} = 0$ . Let  $S_q(1)$  denote the unit sphere in  $T_q M$ . Let  $u$  be a non-negative bounded subharmonic function on  $M$ , that is  $|u(x)| \leq C$ , for all  $x \in M$ . Denote by

$$h(r) = \frac{1}{r^2 \Theta_q(r)} \int_0^r \Theta_q(r).$$

$$(10) \quad \lim_{r \rightarrow \infty} \sup \frac{1}{r^2 \Theta_q(r)} \int_{B(x_0; r)} |u|^p < C^{2p} A(S_q(1)) \lim_{r \rightarrow \infty} h(r) = 0,$$

by Proposition 4.3.

By hypothesis,  $u \Delta u \geq 0$ . Thus, conclusion now follows from (9), (10) and Theorem 4.2.  $\square$

**Corollary 4.5.** *Let  $(M, g)$  be an AHM. Then  $M$  admits Strong Liouville Type Property i.e. if  $u$  is any subharmonic function on  $M$  bounded from above, then  $u$  is a constant function on  $M$ .*

*Proof.* Let  $u$  be a subharmonic function on  $M$  bounded from above. Let  $g(x) = e^{u(x)}$ . Then  $g$  is bounded non-negative subharmonic function on  $M$ , as  $\Delta g = e^{u(x)} (\Delta u + \|\nabla u\|^2) \geq 0$ . Therefore, by Theorem 4.4,  $g$  is a constant function. Consequently,  $u$  is a constant function.  $\square$

**Corollary 4.6.** *Let  $(M, g)$  be an AHM. Let  $u$  be any bounded harmonic function on  $M$ . Then  $u$  is a constant function on  $M$ .*

Li and Wang [15] have proved the Liouville property in the set up of the theorem given below.

**Theorem 4.7.** *Let  $(M, g)$  be a complete manifold satisfying the mean value inequality  $\mathcal{M}(\lambda)$  i.e.  $\mathcal{M}_R(\lambda) \forall R$ . If  $\lambda < 2$  and  $M$  has subexponential volume growth, then bounded harmonic functions are constant.*

*Remark 4.8.* 1) Note that AHM's have subexponential volume growth. We obtain the conclusion of the Theorem 4.7, viz. Corollary 4.6 without assuming the hypothesis of mean value inequality.

2) In [23], Liouville Theorem is obtained for HM (which now we know are flat), by using an Integral Formula for the derivative of harmonic function on a harmonic manifold.

3) Using Liouville property, we prove flatness of AHM in next sections.

## 5. EXISTENCE OF KILLING VECTOR FIELDS

In this section, we show that an AHM  $M$  admits a non-trivial Killing vector field. Using this we show that,  $M$  is flat by a result of [29]. First we quote the splitting result about AH manifolds, define rank of manifold without conjugate points as defined by Knieper (cf [12]), and describe the properties of Killing vector field whose details can be found in [19].

It is also well known that, if a harmonic manifold splits, then it must be a flat manifold ([2]). This result also holds in case of AH manifolds and was proved by A. Zimmer in [29].

**Proposition 5.1.** *Suppose  $(M, g) = (M_1 \times M_2, g_1 \times g_2)$  is a Riemannian product and  $(M, g), (M_1, g_1), (M_2, g_2)$  have no conjugate points. Let  $U^s, U_1^s, U_2^s$ , be the stable Riccati solutions for  $M, M_1, M_2$ , respectively. If  $M$  is AH, then  $M_1, M_2$  are AHM i.e. and  $\text{tr } U^s = \text{tr } U_1^s = \text{tr } U_2^s = 0$ .*

**Definition 5.2. Rank of Manifold without Conjugate Points:** Let  $(M, g)$  be a complete, simply connected and non-compact Riemannian manifold without conjugate points. Let  $V(v) = u^+(v) - u^-(v)$  and correspondingly  $V(t) = V(\varphi^t(v))$  along  $\gamma_v(t)$ . The rank of  $v \in SM$  is defined by

$$\text{rank}(v) = \dim(\ker V(v)) + 1$$

The rank of  $M$  is defined to be

$$\text{rank}(M) = \min\{\text{rank}(v) | v \in SM\}.$$

Note that,  $1 \leq \text{rank}(M) \leq \dim(M)$ .

### 5.1. Killing Vector Fields.

**Definition 5.3.** A vector field  $X$  on a Riemannian manifold  $(M, g)$  is called a *Killing field* if the local flows generated by  $X$  acts by isometries. Equivalently,  $X$  is a Killing field if and only if  $L_X g = 0$ , where  $L$  denotes the Lie derivative of metric  $g$ , with respect to  $X$ .

Now we recall the following results about Killing vector fields. Proofs of the results can be found in [19].

**Proposition 5.4.**  *$X$  is Killing field if and only if  $v \rightarrow \nabla_v X$  is a skew symmetric  $(1, 1)$ - tensor.*

**Proposition 5.5.** *For a given  $p \in M$ , a Killing vector field  $X$  is uniquely determined by  $X(p)$  and  $(\nabla X)(p)$ .*

**Proposition 5.6.** *The set of Killing fields  $\text{iso}(M, g)$  is a Lie algebra of dimension  $\leq \frac{n(n+1)}{2}$ . Furthermore, if  $M$  is complete, then  $\text{iso}(M, g)$  is the Lie algebra of  $\text{Iso}(M, g)$ .*

**Proposition 5.7.** *Let  $X$  be a Killing field on  $(M, g)$  and consider the function  $f = \frac{1}{2}g(X, X) = \frac{1}{2}\|X\|^2$ , then*

- (1)  $\nabla f = -\nabla_X X$ .
- (2)  $\text{Hess } f(V, V) = g(\nabla_V X, \nabla_V X) - R(V, X, X, V)$ .
- (3)  $\Delta f = \|\nabla X\|^2 - \text{Ric}(X, X)$ .

**Lemma 5.8.** *If  $K$  is a Killing field on a Riemannian manifold  $(M, g)$ , then*

$$(11) \quad \nabla_{X,Y}^2 K = -R(K, X)Y$$

We refer [1] for more geometric exposition on Killing vector fields. The following Proposition 5.9 can be found in [1], but we give proof here for the sake of completeness.

**Proposition 5.9.** *A Killing vector field  $X$  on a Riemannian manifold  $(M, g)$  has constant length if and only if every integral curve of the field  $X$  is geodesic.*

*Proof.* For any Killing vector field  $X$  and arbitrary smooth vector field  $Y$  on  $(M, g)$ , we have the following inequalities:

$$\begin{aligned} 0 &= (L_X g)(X, Y) = X.g(X, Y) - g([X, X], Y) - g(X, [X, Y]) \\ &= g(\nabla_X X, Y) + g(X, \nabla_X Y) - g(X, [X, Y]) \\ &= g(\nabla_X X, Y) + g(X, \nabla_Y X) = g(\nabla_X X, Y) + \frac{1}{2}Yg(X, X). \end{aligned}$$

Thus,  $\nabla_X X = 0$  and therefore the required statement follows.  $\square$

## 5.2. Killing Vector Field of Constant Length on Asymptotically Harmonic Manifold With Minimal Horospheres.

**Proposition 5.10.** *Let  $(M, g)$  be an AH manifold of constant  $h \geq 0$ . Then  $\text{Ricci}(v, v) \leq \frac{-h^2}{(n-1)} \leq 0$ , for any  $v \in SM$ , where Ricci denotes the Ricci curvature of  $M$ .*

*Proof.* The conclusion follows by taking the trace of (1), we obtain that on any asymptotically harmonic manifold  $M$ ,

$$(12) \quad -\text{Ricci}(\gamma'_v(t), \gamma'_v(t)) = \text{tr}(u^+)^2(t)$$

for all  $v \in SM$ . By Cauchy-Schwartz inequality,

$$\text{tr}(u^+)^2(t) \geq \frac{(\text{tr } u^+)^2}{(n-1)} = \frac{h^2}{(n-1)}. \text{ Thus, } \text{Ricci}(v, v) \leq -\frac{h^2}{(n-1)} \leq 0. \quad \square$$

The following conclusions follow immediately.

**Corollary 5.11.** *Let  $(M, g)$  be an AH manifold of constant  $h > 0$ . Then  $\text{Ricci}(v, v) \leq \frac{-h^2}{n-1} < 0$ , for any  $v \in SM$ .*

**Corollary 5.12.** *Let  $(M, g)$  be an AHM. Then  $\text{Ricci} \leq 0$ . In fact, by Lemma 3.5, for every  $p \in M$ , there exists a vector  $v \in S_p M$  such that  $\text{Ricci}(v, v) = 0$ .*

*Moreover, proof of Lemma 6.7 (proved in section 6 later) shows that, we can find a small neighbourhood of  $v_0$  say  $V \in SM$  such that, if  $w \in V$ , then  $\text{Ricci}(w(t), w(t)) = 0$ , for all  $w(t) \in \gamma'_w(t)^\perp$ , for all  $t \in (\delta/3, \delta/3)$ , for some small  $\delta > 0$ .*

**Theorem 5.13.** *Let  $(M, g)$  be an AHM. Let  $X$  be a Killing vector field on  $M$ . Then if  $\|X\| < C$ , for some  $C > 0$ , then  $f = \frac{1}{2}g(X, X) = \frac{1}{2}\|X\|^2$  is a constant function. Consequently,  $X$  is parallel and  $\text{Ricci}(X, X) = 0$ . Thus, there exists a non-trivial Killing vector field on  $M$  of constant length.*

*Proof.* Let  $M$  be an AHM. Then by Corollary 5.12,  $\text{Ricci} \leq 0$ . By Proposition 5.7, it follows that  $f$  is a subharmonic function and which is bounded by hypothesis. By Theorem 4.4,  $f$  is a constant function. Hence,  $f$  is harmonic function and therefore,  $\|\nabla X\|^2 = \text{Ricci}(X, X) \leq 0$ , consequently,  $\|\nabla X\|^2 = 0$ . Thus,  $X$  is parallel and hence of constant length and  $\text{Ricci}(X, X) = 0$ . Therefore,  $X$  is solution of ODE with condition  $\|X\| = C > 0$ , which exists. Thus, there exists a non-trivial Killing vector field on  $M$  of constant length.  $\square$

**Corollary 5.14.** *Let  $(M, g)$  be an AH manifold with minimal horospheres. Then the following properties are equivalent:*

- 1)  $X$  is Killing vector field of bounded length and  $\text{Ricci}(X, X) = 0$ .
- 2)  $X$  has constant length.
- 3) The field  $X$  is non-trivial and parallel on  $(M, g)$ .

*Proof.* 1)  $\implies$  2)  $\implies$  3) follows from Theorem 5.13. 3)  $\implies$  2) is obvious on any Riemannian manifold and 3)  $\implies$  1) follows from Theorem 3 of [1]. Thus all the assertions are equivalent.  $\square$

*Remark 5.15.* 1) Corollary 5.14 holds in any manifold of non-positive sectional curvature. In case of AHM, we obtain the same conclusion without the assumption of non-positive curvature.

2) Corollary 5.12 and in turn Theorem 5.13 follow as an application of Borsuk-Ulam theorem, as can be seen from Section 2.

**Lemma 5.16.** *Let  $(M, g)$  be any Riemannian manifold of negative Ricci curvature. Then  $M$  has no non-trivial Killing vector field of constant length.*

*Proof.* Let  $(M, g)$  be a Riemannian manifold with  $\text{Ricci}_M < 0$ . If  $X$  is Killing vector field of constant length, then  $f$  as in Proposition 5.7 is a constant function and hence harmonic function. Therefore,  $X$  is parallel and we have,

$$0 = \Delta f \leq \text{Ricci}(X, X) < 0.$$

This implies that  $X = 0$ . Thus,  $\text{Ricci}(X, X) = 0$  if and only if  $X = 0$ .  $\square$

**Corollary 5.17.** *Let  $(M, g)$  be an AH manifold with  $h > 0$ . Then  $M$  has no non-trivial Killing vector field of constant length.*

*Proof.* Let  $(M, g)$  be an asymptotically harmonic manifold of constant  $h > 0$ . Then by Corollary 5.11,  $\text{Ricci}_M < 0$ . Hence, from Lemma 5.16 the conclusion follows.  $\square$

The proof of the Corollary 5.18 below follows from [19].

**Corollary 5.18.** *Let  $(M, g)$  be an AHM. Then we have  $1 \leq \dim \text{iso}(M, g) \leq \dim M = n$ .*

*Proof.* Since any non-trivial Killing field  $X$  of bounded length is parallel, the linear map:  $X \rightarrow X(p)$  is injective from  $\text{iso}(M, g)$  to  $T_p M$ . And therefore,  $1 \leq \dim \text{iso}(M, g) \leq \dim M = n$ .  $\square$

**Corollary 5.19.** *Let  $(M, g)$  be an AHM. Let  $p = \dim \text{iso}(M, g) \geq 1$ , we have that  $M = \mathbb{R}^p \times N$ .*

*Proof.* On  $M$  there are  $p$  linearly independent parallel vector fields, which we can assume to be orthonormal. As  $M$  is simply connected, each of this vector field is the gradient field for a distance function ( $r : M \rightarrow \mathbb{R}$  is a distance function if  $|\nabla r| = 1$ ). Thus, we have a Riemannian submersion  $M \rightarrow \mathbb{R}^p$  with totally geodesic fibres (Hessian  $\equiv 0$  for distance functions). This gives the desired splitting.  $\square$

Now we show that  $M$  is flat.

**Proposition 5.20.** *Let  $(M, g)$  be an AH manifold of dimension 2. If  $h > 0$ , then  $M$  is isometric to  $\mathbb{H}^2$  of curvature  $-h^2$  and if  $h = 0$ , then  $M$  is flat.*

*Proof.* If  $(M, g)$  is an AH manifold of dimension 2, then by definition  $L$  is a scalar operator i.e.  $Lx = hx$ ,  $h \geq 0$ . By Riccati equation, it follows that  $L^2(x) = -R(x, v)v = -h^2x$ . Thus, if  $h > 0$ , i.e. an AH surface has constant curvature  $-h^2$ . Thus it is  $\mathbb{H}^2$  of curvature  $-h^2$ . If  $h = 0$ , i.e. a surface with all minimal hyperplanes is flat.  $\square$

*Remark 5.21.* 1) Once an existence of non-trivial Killing vector field is proved, AHM  $M^n$ , splits by Corollary 5.19. From Proposition 5.20,  $p \geq 3$  in Corollary 5.19.

2) From a result of [26], any 3 dimensional AHM is flat. Thus, study of AHM starts from dimension  $\geq 4$ .

Now we argue about flatness of  $M$  in the following two different ways.

Argument 1:

**Corollary 5.22.** *Let  $(M, g)$  be an AHM. Then  $(M, g)$  is flat.*

*Proof.* From Corollary 5.19, it follows that  $M = \mathbb{R}^p \times N$ , then from Proposition 5.1, it follows that  $N$  is a AHM and  $\text{Ricci}_N \leq 0$ .  $N$  must be non-compact, as it is complete, simply connected without conjugate points. Since  $\text{Ricci}_N \leq 0$ , again it will have non-trivial parallel Killing fields. And  $N$  will split. Note that  $\text{Ricci } N < 0$  is not possible, as it is an AHM. Continuing this splitting until  $\dim N \leq 3$ , we get that  $M$  is flat.  $\square$

Argument 2.:

**Corollary 5.23.** *Let  $(M, g)$  be an AHM, then  $(M, g)$  is flat.*

*Proof.* From Corollary 5.19, it follows that  $M = \mathbb{R} \times M_1$ . Then from Proposition 5.1, it follows that  $M_1$  is an AH manifold with minimal horospheres of dimension  $n - 1$ . Now we argue by induction on dimension of  $M_1$ . From Proposition 5.20, it suffices to assume that  $\dim M_1 \geq 3$ . If dimension of  $M_1$  is 3, then from [26], it follows that  $M_1$  is flat. Assume the result to be true for dimension  $M_1 = (n - 2)$ . Therefore,  $\dim M = (n - 1)$  and  $M$  is flat. Thus, this shows that an AHM of dimension  $n - 1$  is flat. If dimension of  $M_1$  is  $(n - 1)$ , then as  $M_1$  is an AHM by inductive step it is flat, and thus  $M$  is also flat.  $\square$

Thus, we have proved Main Theorem 1.6 as stated in the introduction. Note that an AHM have apriori subexponential volume growth. Thus we have also shown:

**Theorem 5.24.** *Let  $(M, g)$  be an AH manifold of constant  $h \geq 0$ . Then the following properties are equivalent:*

- 1)  $M$  has subexponential volume growth.
- 2)  $h = 0$ .
- 3)  $M$  is flat.
- 4)  $M$  is of polynomial volume growth.
- 5) Volume entropy of  $M$ ,  $h_{vol}(M) = h$ , is zero.
- 6)  $M$  has rank  $n$ .

The following result was proved in [12]:

**Theorem 5.25.** *Let  $(M, g)$  be an AH manifold of constant  $h \geq 0$  such that  $\|R\| \leq R_0$  and  $\|\nabla R\| \leq R'_0$  with suitable constants  $R_0, R'_0 > 0$ . Then the following properties are equivalent:*

- 1)  $M$  has rank 1.
- 2)  $M$  has Anosov geodesic flow  $\varphi^t : SM \rightarrow SM$ .
- 3)  $M$  is Gromov hyperbolic.
- 4)  $M$  has purely exponential volume growth with growth rate  $h_{vol} = h > 0$ .

**Corollary 5.26.** *Let  $(M, g)$  be an AH manifold of constant  $h \geq 0$ . Then  $M$  has either polynomial volume growth or exponential volume growth.*

*Proof.* Let  $(M, g)$  be an AH manifold of constant  $h \geq 0$ . If  $h = 0$ , then from Theorem 5.24,  $M$  is of polynomial volume growth. If  $h > 0$ , then as  $\lim_{r \rightarrow \infty} \frac{\Theta'_q(r)}{\Theta_q(r)} = h \geq 0$ , for any  $q \in M$ , volume growth is exponential.  $\square$

Thus, we recover the result of [18] that:

**Corollary 5.27.** *Let  $(M, g)$  be a harmonic manifold of constant  $h \geq 0$ . Then  $M$  has either polynomial volume growth or exponential volume growth.*

In [9] among AH Hadamard manifolds, a rigidity theorem with respect to volume entropy, for real, complex and quaternionic hyperbolic symmetric spaces are obtained. In particular, the following result was proved:

**Theorem 5.28.** *Let  $(M, g)$  be an  $n$ -dimensional Hadamard manifold of Ricci curvature  $\text{Ricci} \geq -(n-1)$ . If  $(M, g)$  is AH, then  $h_{\text{vol}} = (n-1)$  and equality holds if and only if  $(M, g)$  is isometric to the real hyperbolic space  $\mathbb{H}^n$  of constant curvature  $-1$ .*

*Remark 5.29.* 1) Our Theorem 5.24 is complementary to Theorem 5.25 and Theorem 5.28.

2) In view of Corollary 5.26 and Corollary 5.27 harmonic manifolds and AH manifolds share the common property viz. that they have the same volume growth.

3) Let  $M$  be an AHM. In [30], it was observed that if  $M$  is without focal points, then  $M$  is flat. Thus, if an AH manifold  $M$  is of non-positive sectional curvature, then  $M$  is flat. We prove the flatness of an AHM without assuming any hypothesis on  $M$ .

## 6. EXISTENCE OF 2, 3 FLATS

In this section, we prove the existence of 2, 3 flats using the results of previous sections. Thus, we reprove the known results that an AHM in dim 2, 3 are flat, by a different method than [26].

*Definition 6.1.* Let  $A$  and  $B$  be two subsets of metric space. The Hausdorff distance between  $A$  and  $B$ , denoted by  $d_H(A, B)$ , is defined by

$$d_H(A, B) = \inf\{r > 0 : A \subset U_r(B) \text{ and } B \subset U_r(A)\}.$$

The next proposition is an important result proved in [19].

**Proposition 6.2.** *Let  $(M, g)$  be a complete, non-compact Riemannian manifold. If  $\gamma_w$  is an asymptote to  $\gamma_v$  from  $p$ , then their Busemann functions are related by*

$$(13) \quad b_v(x) - b_w(x) \leq b_v(p).$$

$$(14) \quad b_v(\gamma_w(t)) = b_v(p) + b_w(\gamma_w(t)),$$

$$(15) \quad = b_v(p) - t.$$

Now we recall few results from [21].

**Corollary 6.3.** *Let  $(M, g)$  be an AH. If  $\gamma_v$  is a geodesic and  $p$  is a point in  $M$ , then the asymptote to  $\gamma_v$  through  $p$  is unique.*

*Proof.* Let  $\gamma_w$  be an asymptote to  $\gamma_v$  from  $p$ .

Then,

$$\langle \nabla b_v(\gamma_w(t)), \gamma_w'(t) \rangle = \frac{d}{dt}(b_v(\gamma_w(t))) = -1 \text{ by (15).}$$

Since  $\nabla b_v$  and  $\gamma_w'(t)$  both are unit tangent vectors, we have,

$$\nabla b_v(\gamma_w(t)) = -\gamma_w'(t).$$

In particular,

$$\nabla b_v(p) = -\gamma_w'(0).$$

This implies that there is only one asymptotic geodesic to  $\gamma_v$  starting from  $p$ , namely in the direction of  $-\nabla b_v(p)$ .  $\square$

**Corollary 6.4.** *Let  $(M, g)$  be an AH manifold. Being asymptotic is an equivalence relation on  $SM$  as well as on the space of all oriented geodesics of  $M$ . Moreover, two oriented geodesics are asymptotic if and only if the corresponding Busemann functions agree upto an additive constant.*

*Proof.* Let  $(M, g)$  be a non-compact AH. Let  $\gamma_w$  be the asymptotic geodesic asymptotic to  $\gamma_v$  starting from  $p$ . Then, clearly  $b_v - b_w$  is a harmonic function. By (13), it attains maximum at  $p$ . Hence, by maximum principle, it is a constant function. This implies that  $b_v(x) - b_w(x) = b_v(p)$ .

Conversely, suppose that  $b_v(x) - b_w(x) = b_v(p)$ . Therefore,

$$\begin{aligned} \nabla b_v(\gamma_w(t)) &= \nabla b_w(\gamma_w(t)) \\ &= -\gamma_w'(t). \end{aligned}$$

Thus,  $\gamma_w$  is an integral curve of  $-\nabla b_v$ . But, from the Corollary 6.3 asymptotes are the integral curves of  $-\nabla b_v$ . Hence, we conclude that  $\gamma_w$  is asymptotic to  $\gamma_v$ . Consequently, it follows that being asymptotic is an equivalence relation on  $SM$ .  $\square$

**Corollary 6.5.** *Let  $(M, g)$  be an AHM. Then any two asymptotic geodesics of  $M$  are bi-asymptotic.*

*Proof.* From Lemma 3.2, it follows that

$$(16) \quad b_v^+ + b_v^- = 0.$$

Let  $\gamma_w^\pm$  be unique asymptote to  $\gamma_v$  starting from  $p$  in positive and negative direction respectively with  $\gamma_w^\pm(0) = p$ .

Therefore, from the above equation,

$$(\nabla b_v^+)(\gamma_w^+(0)) = -(\nabla b_v^-)(\gamma_w^-(0)).$$

It follows that

$$\frac{d}{dt}(\gamma_w^+(t))|_{t=0} = -\frac{d}{dt}(\gamma_w^-(t))|_{t=0}.$$

Thus, two asymptotes  $\gamma_w^\pm$  fit together to form a smooth geodesic  $\gamma_w$ . Hence,  $\gamma_w$  is bi-asymptotic to  $\gamma_v$ .  $\square$

**Corollary 6.6.** *Let  $(M, g)$  be an AHM. Let  $c_1$  and  $c_2$  be two geodesics of  $M$ , such that*

*$d(c_1(t), c_2(t)) \leq l$  as  $t \rightarrow \infty$ . Then,  $c_1$  is bi-asymptotic to  $c_2$ . Thus,  $d(c_1(t), c_2(t)) \leq l$  as  $t \rightarrow -\infty$ , consequently  $d_H(c_1, c_2)$ , Hausdorff distance between two lines is bounded.*

*Proof.* Let  $c_1'(0) = v$  and  $c_2'(0) = w$ . Then, by hypothesis  $b_v^+ - b_w^+ \leq l$ . Clearly,  $b_v^+ - b_w^+$  is a bounded harmonic function. Thus, by Corollary 4.6, Liouville's theorem holds for  $M$ . Therefore,  $b_v^+ - b_w^+ = \text{constant}$ . Hence, from Corollary 6.4,  $c_1$  and  $c_2$  are asymptotic towards  $+\infty$ . Since  $b_v^+ = -b_v^-$  from (16),  $c_1$  and  $c_2$  are bi-asymptotic.  $\square$



### 6.1. Existence of 2 Flat.

**Lemma 6.7.** *Let  $\epsilon > 0$  be given. Then for any point  $p \in M$ , there exists a vector  $v_0 \in S_p M$  such that the Jacobi operator corresponding to  $v_0 \in S_p M$  is zero. That is  $R(x, v_0)v_0 = 0$ , for all  $x \in v_0^\perp$ .*

*Proof.* Fix  $p \in M$ . Let  $v_0 \in S_p M$  be the vector of Lemma 3.5. Hence,  $u^+(v_0) = 0$ . Therefore, given  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$(\text{tr}(u^+(w))^2)^{1/2} = \|u^+(w)\| < \epsilon,$$

whenever  $d^*(v_0, w) < \delta$ , where  $d^*$  is the Sasaki metric on  $SM$ . Hence, in particular,  $|\lambda_{\min}^+(t)| < \epsilon$ , for  $|t| < \delta$ , where  $\lambda_{\min}^+(t)$  is the minimum eigenvalue of  $u^+(t)$ , corresponding to the geodesic line  $\gamma_{v_0}(t)$ . We have  $-\epsilon < \lambda_{\min}^+(t) < \epsilon$ , for all  $|t| < \delta$ . Let  $t_0 \in (0, \delta)$  be the first number such that  $\lambda_{\min}^+(t_0) \neq 0$ . Suppose that  $\lambda_{\min}^+(t_0) > 0$ , then by continuity there exists a small neighbourhood of  $t_0$ , say  $N$ , so that for all  $t \in N$ ,  $\lambda_{\min}^+(t) > 0$ . Consequently,  $u^+(t) = 0$  for all  $t \in N$ , as  $u^+(t)$  is a traceless matrix. (If  $\lambda_{\min}^+(t_0) < 0$ , then a similar argument shows that  $u^+(-t) = 0$  for all  $t \in N_1$ , neighbourhood of  $t_0$ ). And as  $u^+(0) = 0 = u^-(0)$ , it follows that,  $u^+(t) = 0 = u^-(t)$ , for all  $t \in (-\delta/4, \delta/4)$ . Consequently, the Riccati equation implies that  $R(x(t), v(t))v(t) = 0$ . Thus, in particular, the unstable horospheres,  $b_v^{-1}(0)$ , and the stable horospheres,  $b_v^{+1}(0)$ , are totally geodesic at  $t = 0$ .  $\square$

**Lemma 6.8.** *Let  $(M, g)$  be an AHM. For any point  $p \in M$ , there exists two linearly independent vectors,  $X(p), Y(p)$  such that if  $V(p) = \text{span}\{X(p), Y(p)\}$ , then  $\mathcal{R}|_{V(p)} : \Lambda^2 V(p) \rightarrow \Lambda^2 V(p)$  has eigenvalue 0.*

*Proof.* By Theorem 5.13, there exists a killing vector field  $X$  on  $M$  such that  $\|X\| = 1$ . Since  $X$  is a killing vector field on  $M$ , by a result in [19], it follows that

$$\nabla^2_{Y,Z} X = -R(X, Y)Z.$$

Again by Theorem 5.13,  $X$  is a parallel vector field. Hence,  $R(X, Y)Z = 0$ , for any  $Y, Z$ . Consider the vector field  $\nabla b_v$  for any  $v \in S_p M$ . By definition of Killing vector field, it follows that  $\{X(p), \nabla b_v(p)\}$  are linearly independent vectors. Let  $V(p) = \text{span}\{X(p), \nabla b_v(p)\}$ . Then  $\{\nabla b_v(p) \wedge X(p)\}$  forms a basis of  $\Lambda^2 V(p)$ . Consider,

$$\mathcal{R}|_{V(p)} : \Lambda^2 V(p) \rightarrow \Lambda^2 V(p)$$

$$\langle \mathcal{R}(X \wedge Y), V \wedge W \rangle = \langle R(X, Y)W, V \rangle.$$

We have  $R(X, \nabla b_v)\nabla b_v = 0$ . Thus,  $K(X(p), \nabla b_v(p)) = 0$ , where  $K(v, w)$  denotes the sectional curvature of the plane spanned by  $v$  and  $w$ . Hence, the integral curves of  $X$  and  $\nabla b_v$ , geodesics, form coordinate system and around every point  $p \in M$ , we can find a 2-flat.  $\square$

**Corollary 6.9.** *Let  $(M, g)$  and  $V_p$  be as in the above Lemma 6.8. Consider*

$$D : M \rightarrow \cup G_2(T_p M_p),$$

*given by  $p \rightarrow V(p)$ . Then  $D$  is an involutive distribution on  $M$ , whose integral submanifolds are 2-flats. Thus, flat strip theorem holds in  $M$ .*

*Proof.* Let  $v \in S_p M$  and let  $X$  be Killing vector field with  $\|X\| = 1$  and  $X(p) \perp \nabla b_v(p)$ . Consider  $V(p) = \text{span}\{X(p), \nabla b_v(p)\}$ , where  $X$  is a Killing vector field on  $M$  with  $\|X\| = 1$ . Clearly, by Proposition 5.9, integral curves of  $X$  are geodesics. Let  $\alpha$  be the integral curve of  $X$  passing through  $p \in M$  and let  $\gamma_v$  be the integral curves of  $\nabla b_v$  passing through  $p \in M$ . Let  $\beta$  be a geodesic of  $M$  such that  $d_H(\alpha, \beta) \leq C < \infty$ . Then  $\alpha$  and  $\beta$  are bi-asymptotic geodesics by Corollary 6.6 and as they have finite Hausdorff distance and thus bound a strip in  $M$ . We show that this is a flat strip in  $M$ .

Let  $c_{\alpha(t), \beta(t)}(s)$  be the unique unit speed geodesic joining  $\alpha(t)$  and  $\beta(t)$ , that is  $c_{\alpha(t), \beta(t)}(0) = \alpha(t)$ ,  $c_{\alpha(t), \beta(t)}(1) = \beta(t)$ . Consider the parameterized surface defined by

$$(17) \quad \begin{aligned} F : \mathbb{R} \times [0, 1] &\rightarrow M \\ F(t, s) &= c_{\alpha(t), \beta(t)}(s) \end{aligned}$$

Since, geodesics are defined by exponential map, and exponential map is a global diffeomorphism,  $F$  defines an embedding of the rectangular strip  $\mathbb{R} \times [0, 1]$  into  $M$ . Let  $Q = \text{Im} F(t, s)$  with the induced topology from  $M$ . The curves  $s = s_0$  are mapped to all bi-asymptotic geodesics and curve  $t = t_0$  is mapped to unit speed geodesic  $c_{\alpha(t_0), \beta(t_0)}(s)$ . By the first variation formula of arc length,

$$d(\alpha(t), \beta(t)) = \int_0^1 \|c'_{\alpha(t_0), \beta(t_0)}(s)\| ds = 1, \quad \forall t \in \mathbb{R}.$$

Therefore,  $H(\alpha, \beta) = 1$ .

From Lemma 6.8, it follows that  $K(v, w) = 0$ , for any  $v, w$  tangent and normal vectors, respectively, along any bi-asymptotic geodesic lying in  $Q$ . Therefore,  $Q$  is flat and  $F$  is totally geodesic embedding. Thus,  $D$  is an involutive distribution on  $M$ , with integral submanifold  $Q$ , a 2 flat in  $M$ .  $\square$

*Remark 6.10.* From Proposition 5.20, an AHM of dimension 2 is flat. We also recover this result from Lemma 6.8 and Corollary 6.9.

**Corollary 6.11.** *Let  $(M, g)$  be an AHM of dimension 2. Then  $M$  is flat.*

**Corollary 6.12.** *Let  $(M, g)$  be an AHM. Then  $(M, g)$  is flat.*

*Proof.* From Corollary 6.9 and Lemma 6.8, it follows that  $M = \mathbb{R}^2 \times N$ . Then by earlier argument, it follows that  $M$  is flat.  $\square$

*Remark 6.13.* Let  $(M, g)$  be an AHM. Let  $X$  be a killing vector field on  $M$  with  $\|X\| = 1$  and let  $v_0$  be the vector as in the Lemma 3.5. Let  $X(t)$  denote the Killing vector field along  $\gamma_{v_0}(t)$ . The vectors  $X(t), v(t) = \gamma'_{v_0}(t)$  are linearly independent. Let  $Y(t) = X(t)\cos t + \gamma'_{v_0}(t)\sin t$ , be the unit vector. Consider the function

$$f : \mathbb{R} \rightarrow \mathbb{R},$$

given by

$$f(t) = \text{Ricci}(v_t, v_t).$$

Then  $f(t) \leq 0$ .  $f$  attains maximum at  $t = 0, t = \frac{\pi}{2}$ , and  $t = n\pi$ , where  $n$  is an even positive integer. Therefore, it follows that  $f''(t) \leq 0$ , i.e.  $f$  is a

concave function. Mean value theorem then shows that  $f \equiv 0$ . Therefore, the strip bounded by integral curve of  $X(t)$  and geodesic  $\gamma_{v_0}(t)$  spans a flat strip.

## 6.2. Existence of 3 Flat.

**Lemma 6.14.** *Let  $(M, g)$  be an AHM. For any point  $p \in M$ , there exists three linearly independent vectors,  $X(p), Y(p), Z(p)$  such that if  $W(p) = \text{span}\{X(p), Y(p), Z(p)\}$ , then  $\mathcal{R}|_{W(p)} : \Lambda^2 W(p) \rightarrow \Lambda^2 W(p)$  is a diagonal operator with all eigenvalues 0.*

*Proof.* By Lemma 6.7, for every point  $p \in M$  there exists a vector  $e_1 = \nabla b_{v_0} \in S_p M$  such that  $R(x, e_1)e_1 = 0, \forall x \in e_1^\perp$ . Also by Theorem 5.13, there exists a killing vector field  $X$  on manifold such that  $\|X\| = 1$ . By definition of Killing vector field, it follows that  $X(p), e_1$  are linearly independent vectors. We may assume that  $X(p), e_1$  are orthogonal. Consider the subspace  $W(p)$  of  $T_p M$  spanned by  $\{e_1, X(p), e_3\}$ , where  $e_3$  is an unit vector orthogonal to  $\{X(p), e_1\}$ . From Lemma 6.7, it follows that  $R(X(p), e_1)e_1 = 0, R(e_3, e_1)e_1 = 0$ . Also as  $X(p)$  is a killing vector field,  $R(X, Y)Z = 0$ . Hence,  $R(e_1, X(p))X(p) = 0, R(e_3, X(p))X(p) = 0$ . Therefore,  $K(e_1, X(p)) = 0, K(e_1, e_3) = 0, K(e_3, X(p)) = 0$ . Then  $\{e_1 \wedge X(p), e_1 \wedge e_3, X(p) \wedge e_3\}$  forms a basis of  $\Lambda^2 W(p)$ . We want to show that the operator, considered as map

$$\mathcal{R}|_{W(p)} : \Lambda^2 V(p) \rightarrow \Lambda^2 W(p),$$

$$\langle \mathcal{R}(X \wedge Y), V \wedge W \rangle = \langle R(X, Y)W, V \rangle$$

is diagonal in this basis.

From Lemma 6.7 and Theorem 5.13,  $\mathcal{R}(e_1 \wedge e_3) \perp \text{span}\{e_1 \wedge X(p), X(p) \wedge e_3\}$ ,  $\mathcal{R}(e_1 \wedge X(p)) \perp \text{span}\{e_1 \wedge e_3, X(p) \wedge e_3\}$  and  $\mathcal{R}(X(p) \wedge e_3) \perp \text{span}\{e_1 \wedge X(p), e_1 \wedge e_3\}$ . Hence,  $\mathcal{R}|_{W(p)}$  is diagonal in the basis with all the eigenvalues zero. Hence, the integral curve of  $X$  and  $\nabla b_{v_0}, \gamma_{v_0}$ , geodesics, and the geodesic adapted to  $e_3$ , form coordinate system around every point  $p \in M$ . Hence, we can find a 3-flat in  $M$ .  $\square$

The proof of the Corollary 6.15 below is analogous to the proof of the Corollary 6.9.

**Corollary 6.15.** *Let  $(M, g)$  and  $W_p$  be as in the above Lemma 6.14. Consider*

$$D : M \rightarrow \cup G_3(T_p M_p),$$

*given by  $p \rightarrow W(p)$ . Then  $D$  is an involutive distribution on  $M$ , whose integral submanifolds are 3-flats.*

Thus, from Lemma 6.14 and Corollary 6.15, we recover the known result proved in [26].

**Corollary 6.16.** *Let  $(M, g)$  be an AHM in dimension 3. Then  $M$  is flat.*

**Corollary 6.17.** *Let  $(M, g)$  be an AHM. Then  $(M, g)$  is flat.*

*Proof.* From Lemma 6.14 and Corollary 6.15, it follows that  $M = \mathbb{R}^3 \times N$ . Thus,  $M$  is flat by an earlier argument.  $\square$

*Remark 6.18.* We shortly describe the proof of the theorem in [23] that harmonic manifolds with minimal horospheres are flat.

The original proof of Theorem 1.4 given in [23] is based on the idea of the Szabo's proof of the Lichnerowicz's conjecture in compact case.

Using a result of P. Li and J. Wang, it was shown that  $\text{span}\{b_v^2\}$ , where  $b_v$  is a Busemann function for  $v$  (defined in section 2), is a finite dimensional vector space. Then  $b_v^2$  was averaged (idea which can be employed only for harmonic manifolds), and a parallel displaced family,  $g_\gamma$ , of real valued functions on  $\mathbb{R}$ , for every geodesic  $\gamma$  was obtained.  $\text{Span}\{g_\gamma\}$  is also a finite dimensional vector space and therefore, the generator function  $g$  is a trigonometric polynomial. By using properties of  $g$ ,  $g$  was written in the simpler form. Then another family of radial functions  $\mu_\gamma$  was introduced. The generator function,  $\mu$  was obtained by generalizing co-ordinate function  $r \cos \theta$  on a harmonic manifold. Then using properties of  $\mu$ , the two families were related. It was observed that  $g$  and  $\mu$  are almost periodic functions. Finally, using the Characteristic Property of an almost periodic function, it was proved that  $M$  is Ricci flat.

The above sketch of the proof reveals that the methods of the above proof uses harmonicity of  $M$  heavily, while our proof obtained in this paper for AH manifolds is wider, in comparison.

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